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Minimal Machines with Several Initial States are Not Unique*

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It is shown that there is an infinite number of reduced deterministic finite machines, all of which compute the same input output relation.

Any two minimal machines $\mathscr{M}$ and $\mathscr{M}'$ computing the same function $f: \Sigma^* \to \Gamma^*$ are isomorphic (Eilenberg, 1974). We show that this theorem depends heavily on the fact that $\mathscr{M}$ and $\mathscr{M}'$ have only one initial state.

As defined by Eilenberg (1974), a complete sequential machine consists of the following data: a finite set Q of states, an initial state $i \in Q$ and a function $Q \times \Sigma \to Q \times \Gamma$, where Σ and Γ are finite alphabets. This function breaks up into a function $Q \times \Sigma \to Q$, $(q, \sigma) \mapsto q\sigma$, which is called the next state function, and into the output function $\lambda: Q \times \Sigma \to \Gamma$. As usual these functions are extended to $Q \times \Sigma^* \to Q$ and $\lambda: Q \times \Sigma^* \to \Gamma^*$ by the inductive definitions $q1 = q$, $q(s\sigma) = (qs)\sigma$, $(q, 1)\lambda = 1$ and $(q, s\sigma)\lambda = (q, s)\lambda(qs, \sigma)\lambda$, where Σ^* and Γ^* are free monoids with unit element 1 over Σ and Γ, respectively, and $q \in Q$, $s \in \Sigma^*$, $\sigma \in \Sigma$ are arbitrary elements.

Since we are interested in complete sequential machines with several initial states, we define $\mathscr{M} = (Q, I, \lambda)$ to be a machine, if the set Q, the next state function and the output function λ are defined as above, and $I \subset Q$ is a set of initial states. The function computed by a state $q \in Q$ is the function $f_q: \Sigma^* \to \Gamma^*$, defined by $sf_q = (q, s)\lambda$. If $Q_1 \subset Q$ is a set of states, f_{Q_1} denotes the relation, whose graph $\# f_{Q_1} \subset \Sigma^* \times \Gamma^*$ is the following union of graphs: $\# f_{Q_1} = \cup \{\# f_q \mid q \in Q_1\} = \{(s, t) \mid sf_q = t, q \in Q_1\}$. The relation computed by $\mathscr{M}$ is the relation $f_{\mathscr{M}} = f_I$.

An initial state $i \in I$ is called redundant, if $f_I = f_{I_1}$ for $I_1 = I - \{i\}$ and $\mathscr{M}$ is irredundant if I contains no redundant initial states. In accordance with the usual definition, two states q and q' are called equivalent if they compute the

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same function $f_q = f_{q'}$. $\mathscr{M}$ is reduced, if no two distinct states in $\mathscr{M}$ are equivalent. $\mathscr{M}$ is accessible or initially connected, if $I\Sigma^* = \{is \mid i \in I, s \in \Sigma^*\} = Q$ and $\mathscr{M}$ is recurrent or strongly connected if $q\Sigma^* = \{qs \mid s \in \Sigma^*\} = Q$ for every state $q \in Q$. $\mathscr{M}$ is called minimal, if $\mathscr{M}$ is reduced and accessible.

If $f: \Sigma^* \to \Gamma^*$ is a relation and $s \in \Sigma^*$, $g \in \Gamma^*$ are words, then the relation $f(s, g): \Sigma^* \to \Gamma^*$ is defined by its graph:

$$\#f(s, g) = \{(ss', gg') \mid (s', g') \in \#f\}.$$

THEOREM. *There is an infinite number of (nonisomorphic) minimal machines, which are irredundant and compute the same relation $f: \Sigma^* \to \Gamma^*$.*

Proof. Let $\Sigma = \{\sigma_1, \sigma_2\}$ and $\Gamma = \{\gamma_1, \gamma_2\}$ be alphabets with two letters. For all natural numbers $n > 1$, we define machines $\mathscr{M}_n = (Q_n, I_n, \lambda_n)$, Q_n having $4n$ states denoted by $p_i, q_j, r_k, s_l, t_1, t_2$, $(1 \leqslant i, l \leqslant n, 1 \leqslant j, k < n)$. The next state and output function are given by Table I. For all machines the set of initial states is $I_n = \{p_1, s_1\}$.

TABLE I

Machine $\mathscr{M}_n$

state q	p_1	p_2	...	p_{n-1}	p_n	q_1	q_2	...	q_{n-1}	t_1
$q\sigma_1$	q_1	q_2	...	q_{n-1}	p_n	p_2	p_3	...	p_n	t_1
$q\sigma_2$	t_1	t_1	...	t_1	t_1	t_2	t_2	...	t_2	t_1
$(q, \sigma_1)\lambda$	γ_1	γ_1	...	γ_1	γ_1	γ_1	γ_1	...	γ_1	γ_2
$(q_1, \sigma_2)\lambda$	γ_2	γ_2	...	γ_2	γ_2	γ_2	γ_2	...	γ_2	γ_1
state q	s_1	s_2	...	s_{n-1}	s_n	r_1	r_2	...	r_{n-1}	t_2
$q\sigma_1$	r_1	r_2	...	r_{n-1}	s_n	s_2	s_3	...	s_n	t_2
$q\sigma_2$	t_2	t_2	...	t_2	t_2	t_1	t_1	...	t_1	t_2
$(q, \sigma_1)\lambda$	γ_1	γ_1	...	γ_1	γ_1	γ_1	γ_1	...	γ_1	γ_1
$(q, \sigma_2)\lambda$	γ_2	γ_2	...	γ_2	γ_2	γ_2	γ_2	...	γ_2	γ_1

Each $\mathscr{M}_n$ is accessible. We first prove that $\mathscr{M}_n$ is reduced. Clearly, t_1 and t_2 are not equivalent and both are not equivalent to any other state. From this follows that the pairs (p_i, q_i), (q_j, r_k), (r_k, s_l), (s_l, p_i) denote nonequivalent states. Next we consider a pair (p_{i_1}, p_{i_2}) of different states, where $1 \leqslant i_1$, $i_2 \leqslant n$ and $i_1 < i_2$. From $p_{i_1}\sigma_1^{2(n-i_2)+1}\sigma_2 = t_2$ and $p_{i_2}\sigma_1^{2(n-i_2)+1}\sigma_2 = t_1$ it follows, that p_{i_1} and p_{i_2} are not equivalent. Since $(q_{j_1}\sigma_1, q_{j_2}\sigma_1) = (p_{j_1+1}, p_{j_2+1})$ are different, if q_{j_1} and q_{j_2} are different, also q_{j_1} and q_{j_2} cannot be equivalent. Finally, for p_i and r_k the same is shown be the observation, that the equalities $p_i\sigma_1^{2(n-1)}\sigma_2 = t_1$ and $r_k\sigma_1^{2(n-1)}\sigma_2 = t_2$ are satisfied. By the symmetry of the state graph of $\mathcal{M}_n$, in all remaining cases the same arguments can be applied. Thus $\mathcal{M}_n$ is minimal.

For all $n > 1$ the graph of $f_{\mathcal{M}_n}$ is given by the union of graphs $\#f_{t_1}(\sigma_1{}^i\sigma_2, \gamma_1{}^i\gamma_2)$ and $\#f_{t_2}(\sigma_1{}^i\sigma_2, \gamma_1{}^i\gamma_2)$, where i runs through the set of nonnegative natural numbers. Hence all relations $f_{\mathcal{M}_n} = f$ are identical.

Since the relation f is not a function and $\mathcal{M}_n$ has exactly two initial states, each $\mathcal{M}_n$ is irredundant.

We now show that minimal machines computing the same relation are isomorphic, if they are recurrent. Machines $\mathcal{M} = (Q, I, \lambda)$ and $\mathcal{M}' = (Q', I', \lambda')$ are isomorphic, if there is a bijection $\varphi: Q \to Q'$ satisfying $(q\varphi)\,\sigma = (q\sigma)\,\varphi$ and $(q\varphi, \sigma)\,\lambda' = (q, \sigma)\,\lambda$ for all $q \in Q$ and $\sigma \in \Sigma$. Note that initial states are not involved in this definition.

THEOREM. *Any two minimal, irredundant and recurrent machines* $\mathcal{M} = (Q, I, \lambda)$ *and* $\mathcal{M}' = (Q', I', \lambda')$ *computing the same relation* $f_{\mathcal{M}} = f_{\mathcal{M}'}$ *are isomorphic.*

Proof. Let $i_0 \in I$ be an initial state of $\mathcal{M}$. Since $\mathcal{M}$ is irredundant, there is a word $s \in \Sigma^*$ such that the outputs $(i_0, s)\,\lambda$ and $(i, s)\,\lambda$ differ for all $i \in I$ with $i \neq i_0$.

Since $\mathcal{M}$ and $\mathcal{M}'$ compute the same relation, $(i_0', s)\,\lambda' = (i_0, s)\,\lambda$ holds for some initial state $i_0' \in I'$ of $\mathcal{M}'$. We prove that $p = i_0 s$ and $p' = i_0' s$ compute the same function $f_p = f_{p'}$.

If $t \in \Sigma^*$ is a word, then $(st, (i_0', st)\,\lambda') \in \#f_{\mathcal{M}'} = \#f_{\mathcal{M}}$ implies $(i, st)\,\lambda = (i_0', st)\,\lambda'$ for some $i \in I$. Since $(i, s)\,\lambda = (i_0', s)\,\lambda = (i_0, s)\,\lambda$ we can conclude $i = i_0$ and $(i_0, s)\,\lambda\,(i_0 s, t)\,\lambda = (i_0, st)\,\lambda = (i_0', st)\,\lambda' = (i_0', s)\,\lambda'(i_0's, t)\,\lambda'$. Therefore $tf_p = (p, t)\,\lambda = (p', t)\,\lambda' = tf_{p'}$.

The machines $\mathcal{M}_1 = (Q, I_1, \lambda)$ and $\mathcal{M}_1' = (Q', I_1', \lambda')$, which differ from $\mathcal{M}$ and $\mathcal{M}'$ only by the sets of initial states $I_1 = \{p\}$ and $I_1' = \{p'\}$ are minimal, since $\mathcal{M}$ and $\mathcal{M}'$ are minimal and recurrent. Thus by (Eilenberg, 1974, Chap. XII, Theorem 4.1), the machines $\mathcal{M}_1$ and $\mathcal{M}_1'$ are isomorphic, since they compute the same function $f_p = f_{p'}$. Thus $\mathcal{M}$ and $\mathcal{M}'$ are also isomorphic.

The first theorem shows that minimal machines with several initial states are not necessarily minimal with respect to the number of states. But since the relation computed by a machine can be considered as a rational set over the alphabet $\Sigma \times \Gamma$, it is decidable whether two machines $\mathscr{M}$ and $\mathscr{M}'$ compute the same relation. Therefore for any machine $\mathscr{M}$ a machine $\mathscr{M}'$ can be constructed, which computes the same relation and which is minimal with respect to the number of states.

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